13.4 Acceleration and the TNB-Frame

Entry Task

Consider $r(t) = \langle t, t^2, 5 \rangle$.

Find the unit tangent vector **T**(t).

Thm: T and **T'** are always orthogonal.

Proof:

Since $\boldsymbol{T} \cdot \boldsymbol{T} = |\boldsymbol{T}|^2 = 1$,

we differentiate both sides to get

$$\boldsymbol{T}' \cdot \boldsymbol{T} + \boldsymbol{T} \cdot \boldsymbol{T}' = 0.$$

So $2\boldsymbol{T} \cdot \boldsymbol{T}' = 0.$

Thus, $\boldsymbol{T} \cdot \boldsymbol{T}' = 0.$ (QED)

The TNB-Frame

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} = \text{unit tangent}$$

$$\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{|\boldsymbol{T}'(t)|} = \text{principal unit normal}$$

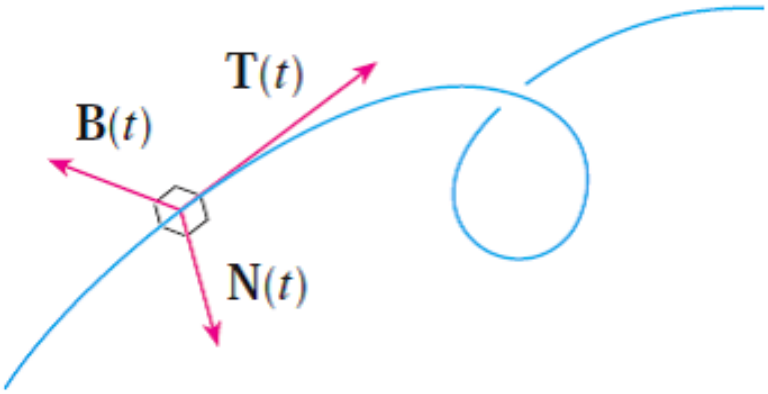
$$\boldsymbol{B}(t) = \boldsymbol{T}(t) \times \boldsymbol{N}(t) = \text{binormal}$$

Note: These vectors are very interesting, but are a bit of mess to compute by hand. Think about using this on our entry task and what would happen.

TNB-Frame Facts:

$\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$ together give a good approximation of the "plane of motion" called the *osculating (kissing) plane.*

$\boldsymbol{T}(t)$, $\boldsymbol{N}(t)$, $\boldsymbol{r}'(t)$, and $\boldsymbol{r}''(t)$ are ALL parallel to the osculating plane



Desmos: https://desmos.com/3d/lucgz79ggg

13.4 Position, Velocity, Acceleration

If ***t = time*** and position is given by

$$\boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle$$

then

$$\boldsymbol{r}'(t) = \lim_{h \to 0} \frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h} = \text{velocity}$$

$$|\boldsymbol{r}'(t)| = \frac{\text{change in dist}}{\text{change in time}} = \text{speed} = \frac{\text{ds}}{\text{dt}}$$

$$\boldsymbol{r}''(t) = \lim_{h \to 0} \frac{\boldsymbol{r}'(t+h) - \boldsymbol{r}'(t)}{h}$$

$$= \frac{\text{change in velocity}}{\text{change in time}} = \boldsymbol{a}(t)$$

Example

Let *t* be time in seconds and assume the position of an object (in feet) is given by

$$r(t) = \langle 50t, 6 + 50t - 16t^2 \rangle$$

Compute $\boldsymbol{r}'(t)$, $|\boldsymbol{r}'(t)|$, and $\boldsymbol{r}''(t)$.

Visual: https://desmos.com/calculator/7e3d6db11e

HUGE application: ANY motion problem

Newton's 2nd Law of Motion states

$$\boldsymbol{F} = m \cdot \boldsymbol{a} \text{ , so } \boldsymbol{a} = \frac{1}{m} \cdot \boldsymbol{F}$$

- If $\boldsymbol{F} = \langle 0,0,0 \rangle$, then all the forces 'balance out' and there is no acceleration.
- If $\boldsymbol{F} \neq \langle 0,0,0 \rangle$, then acceleration occurs and we integrate to find velocity and position.

HW Example

An object of mass 2 kg is being acted on by the force

$\boldsymbol{F} = \langle 12t, 2e^{t}, 9e^{-3t} \rangle$.

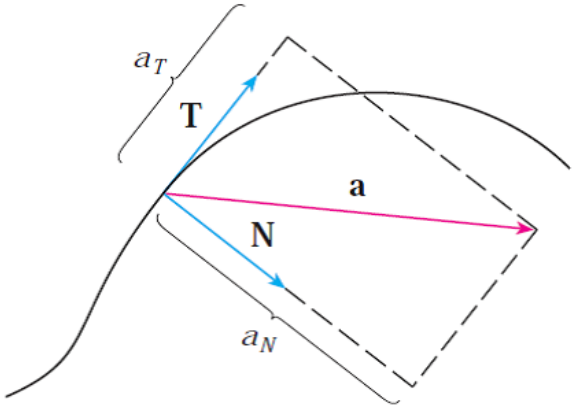
You are given

$\boldsymbol{v}(0) = \langle 0, -8, -3 \rangle$ and $\boldsymbol{r}(0) = \langle 1, 3, 7 \rangle$.

Find the position function.

What direction does the acceleration point?

$$a = a_{\mathrm{t}}T + a_{\mathrm{n}}N$$



Tangential Component Facts

Def′n: $a_T = \mathrm{comp}_{\boldsymbol{T}}(\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{T}$

Shortcut: $a_T = \dfrac{\boldsymbol{r}' \cdot \boldsymbol{r}''}{|\boldsymbol{r}'|}$

Interpretation: $a_T = v' = \dfrac{d}{dt}|r'(t)|$

$=$ "deriv. of speed"

Note: a_T can be positive or negative (or zero)

positive – speedometer increasing

negative – speedometer decreasing

Normal Component Facts

Def′n: $a_N = \mathrm{comp}_{\boldsymbol{N}}(\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{N}$

Shortcut: $a_N = \dfrac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|}$

Interpretation: $a_N = kv^2$

$=$ curvature · (speed)^2

Note: a_N is always positive (or zero)

(accel. points "inward" relative to the curve, but not always "directly" inward)

Example

$$\vec{\boldsymbol{r}}(t) = < \cos(t) , \sin(t) , t >$$

Find the tangential and normal components of acceleration.

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Consider the position vector

$\boldsymbol{r}(t) = \langle 5t, e^{t}, e^{-3t} \rangle$.

Find all values of t at which the tangential component of acceleration is zero.

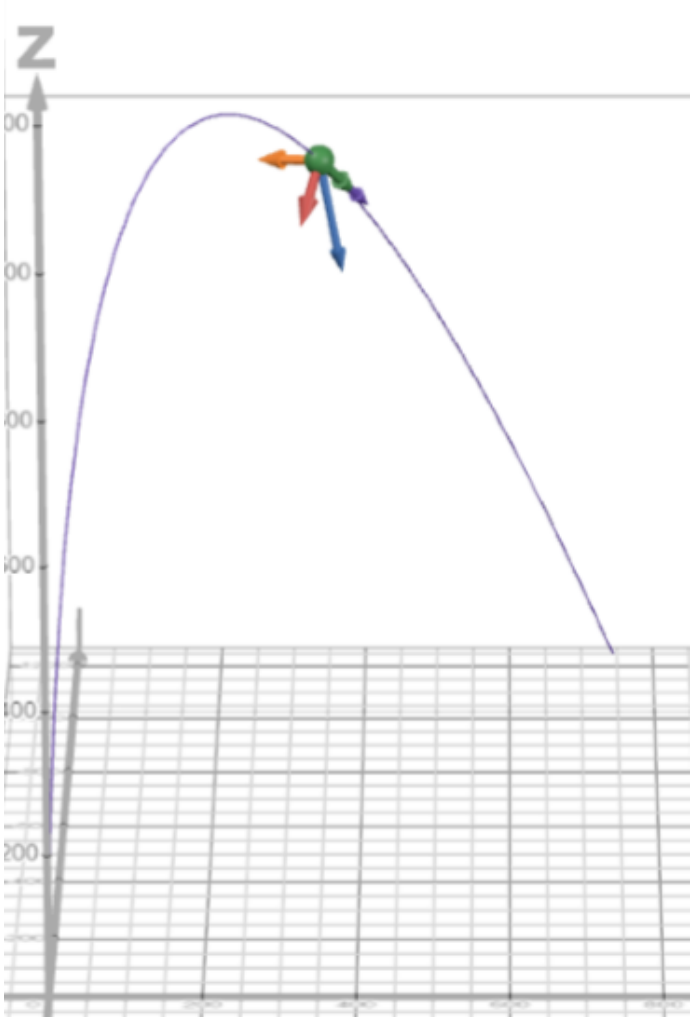
Two Fun Examples (that we will skip for time)

We throw a ball from the origin with an initial velocity parallel to the yz-plane. There are two forces, a force due to gravity and a force due to wind given by

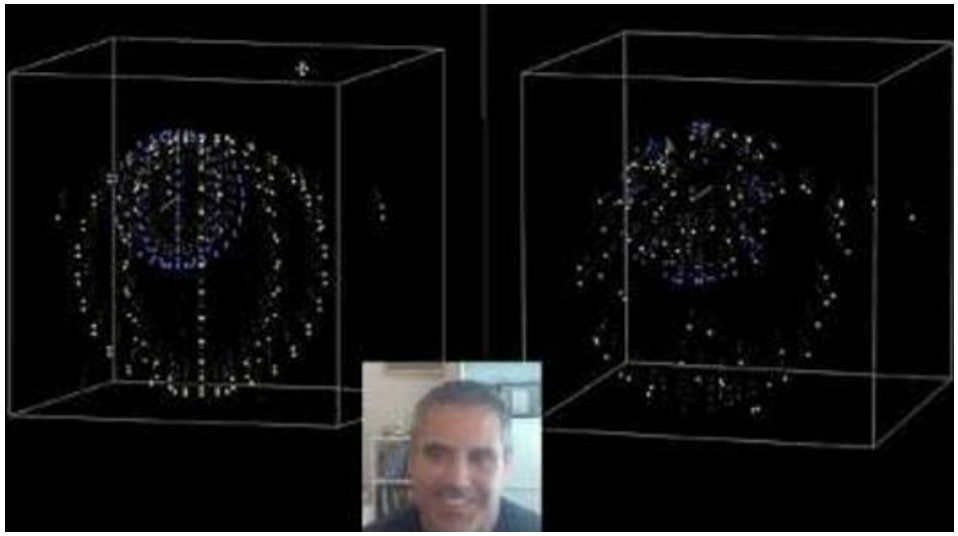
$$\boldsymbol{F}_g = \langle 0, 0, -mg \rangle$$

$$\boldsymbol{F}_w = \langle mw, 0, 0 \rangle$$

Visual: https://desmos.com/3d/875cbc9d5a



Projectile Motion in 3D and fireworks



If we assume an object is accelerating in 3D due to gravity only (no air resistance or other forces), then we can model the motion by integrating the second derivative vector ⟨0, 0, −g⟩.

$$\boldsymbol{r}(t) = r_0 + v_0 t + \tfrac{1}{2}\langle 0, 0, -g\rangle t^2$$

where $\mathbf{r_o}$ is the initial position and $\mathbf{v_o}$ is the initial velocity.

$$\boldsymbol{v}_0 = \langle v_1, v_2, v_3\rangle$$

$$x(t) = x_0 + v_1 t$$

$$y(t) = y_0 + v_2 t$$

$$z(t) = z_0 + v_3 t - \tfrac{1}{2} g t^2$$

Visual: https://www.desmos.com/3d/qkrqow2wuu

Derivation of a_T and a_N interpretations

(you can skip this)

Note visually that you can see:

$$\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$$

We are trying to find the numbers in front of **T** and **N.**

Let $v(t) = |\vec{\boldsymbol{v}}(t)|$ = speed.

$$\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}\boldsymbol{\prime}(t)}{|\vec{\boldsymbol{r}}\boldsymbol{\prime}(t)|} = \frac{\vec{\boldsymbol{v}}(t)}{v(t)} \text{ implies } \vec{\boldsymbol{v}} = v\vec{\boldsymbol{T}}.$$

$$\kappa(t) = \frac{|\vec{\boldsymbol{T}}\boldsymbol{\prime}(t)|}{|\vec{\boldsymbol{r}}\boldsymbol{\prime}(t)|} = \frac{|\vec{\boldsymbol{T}}\boldsymbol{\prime}|}{v(t)} \text{ implies } |\vec{\boldsymbol{T}}\boldsymbol{\prime}| = \kappa v.$$

$$\vec{\boldsymbol{N}}(t) = \frac{\vec{\boldsymbol{T}}\boldsymbol{\prime}(t)}{|\vec{\boldsymbol{T}}\boldsymbol{\prime}(t)|} = \frac{\vec{\boldsymbol{T}}\boldsymbol{\prime}}{\kappa v} \text{ implies } \vec{\boldsymbol{T}}\boldsymbol{\prime} = \kappa v\vec{\boldsymbol{N}}.$$

Differentiating the first fact above gives

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{v}}\boldsymbol{\prime} = v'\vec{\boldsymbol{T}} + v\vec{\boldsymbol{T}}\boldsymbol{\prime}, \text{ so}$$

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{v}}\boldsymbol{\prime} = v'\vec{\boldsymbol{T}} + kv^2\vec{\boldsymbol{N}}.$$